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COLIN MACLAURIN. A. M.

Late Professor of Mathematics in the University of Edinburgh, & Fellow of the Royal Society.

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Volume XXVII Number 3

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Development of Mathematics in Secondary Schools of the United States

By F. L. WREN

George Peabody College for Teachers, Nashville, Tennessee

and

H. B. MCDONOUGH

Culleoka, Tennessee

PART I

MATHEMATICS IN THE LATIN GRAMMAR SCHOOL*

THE EARLY SETTLERS of the American colonies were influenced by the educational principles and practices of the countries from which they had come, and these educational policies had, in turn, evolved under the influence of the Renaissance and Reformation. These two movements had co-operated in fostering a new liberal education which "implied the intellectual, moral, esthetic, and physical development of the individual for true fulfillment of complete life as a citizen."[1] This, under the influence of the times,

* The second part of this discussion will appear in the April issue of THE MATHEMATICS TEACHER.

[1] I. L. Kandel, *History of Secondary Education*, Houghton-Mifflin Co. (1930), p. 65.

meant a linguistic type of education, the language being Latin. The curricula of the schools were thus organized to conform to the objectives which this ideal fostered. Various formulations of these objectives may be found in the literature of the time.

> Practically all of them agree in including the elements of knowledge, piety, and eloquence. The first indicated an ability to read, write, and speak the Latin language and perhaps a knowledge of the content of Latin literature; by piety was meant a familiarity with the scriptures, the catechism, credal forms, and ecclesiastical ceremonies, all gained more or less directly through the school; by eloquence was meant an ability to use the Latin language effectively in public activities.[2]

These schools, which flourished in England and on the continent during the sixteenth and seventeenth centuries became the immediate prototypes of the American colonial grammar schools.

The primary purpose of the Latin grammar schools, the first one of which was established in Boston in 1635, was to train boys in Latin grammar and literature for admission to college where they might prepare themselves for the different professions and especially the ministry. Under such a program the study of numbers was in general overlooked. Brown[3] states that "it was no uncommon thing to find scholars almost ready for the university who were unable to make out the number of pages, chapters, or other divisions in the books they were reading." The curriculum of this colonial secondary school was quite limited and only a few instances are found where legislation made provisions for courses other than those in Latin. In Clew's *Educational Legislation and Administration of Colonial Governments* only nine references are made to the teaching of mathematics. In the *Governor's Journal of Boston* of 1645 we find that "divers free schools were erected, as at Roxbury and at Boston, where they made an order to allow forever fifty pounds to the Master and a house, and thirty pounds to the usher who should also teach to read, write, and cipher." From the *New Amsterdam Records* we learn that Jacob Van Corlaar had taught reading, writing, and ciphering (1658).[4] A law was passed in Boston, in 1682, authorizing the selectmen to establish one or more "free

[2] Paul Monroe, *Principles of Secondary Education*, The Macmillan Co. (1915), p. 38.

[3] E. E. Brown, *The Making of Our Middle Schools*, Longmans, Green and Co. (1903), p. 20.

[4] E. W. Clews, *Educational Legislation and Administration of Colonial Governments*, Columbia University Press, (1899), p. 207.

schools to teach children to write and cipher"[5] The court of Hartford, Connecticut, ordered, in 1690, "that two free schools be kept and maintained for the use of all such children as shall come there, after they can first read the psalter, to teach such reading, writing, arithmetic, the Latin and Greek tongues; the one at Hartford, the other at New Haven."[6] From a later part of the *New Amsterdam Records* we learn that George Muirson was authorized, in 1703, to instruct in the art of arithmetic and that Andrew Foucautt was employed to teach arithmetic in the English and French schools within the city of New York. Two years later, Henry Lindly was employed to teach English and Latin and also arithmetic in the town of Jamaica, N. Y. In 1712 Allane Jarrott obtained license from Governor Hunter "to teach writing, arithmetic, navigation, and other parts of mathematics" to all persons within the city or province of New York.[7] In 1719, an act for the encouragement of learning was passed by the Assembly of South Carolina which read, in part, as follows:

> John Douglas, shall be, and hereby is declared to be Master of a Grammar School in Charlestown, for teaching the Greek and Latin languages, and shall choose one usher to the said school who is empowered and required to assist the master aforesaid in teaching the languages, reading, English, writing, arithmetic or such other parts of mathematics as he is capable to teach.[8]

In 1729, Samuel Brown made a grant of one hundred and twenty pounds to the Grammar School of Salem, Massachusetts, to be used toward the education of eight or ten poor scholars "yearly in the Grammar learning or the mathematics, viz: the mariner's art."[9] In New York in 1732 "a Free School for teaching the Latin and Greek, and practical branches of mathematics was incorporated by law."[10] At a council meeting held in Philadelphia, the 2nd of June, 1759, to protest against legislative action which seemed to be hostile to the best interests of the College, Academy, and Charitable School of that city, it was revealed that arithmetic was a part of the curriculum of the secondary school of the institution and that Euclid was being taught in the College. These facts were offered in partial justification of the existence of the school.[11]

[5] Henry Barnard, *The American Journal of Education*, Vol. 1 (1858), p. 301.
[6] E. W. Clews, *op. cit.*, p. 96.
[7] *Ibid.*, pp. 235–238.
[8] *Ibid.*, p. 457.
[9] Henry Barnard, *op. cit.*, p. 320.
[10] *Ibid.*, p. 300.
[11] E. W. Clews, *op. cit.*, p. 309.

Up to the beginning of the nineteenth century we find only three subjects required for entrance to college, namely, Latin, Greek, and Arithmetic. Mathematics, to the extent of elementary arithmetic, appeared among the subjects for entrance to Yale College in 1745. The Trustees of Princeton University voted, for the first time in 1760, that all who were to be admitted to the freshman class should be "acquainted with vulgar arithmetic." In 1794, however, arithmetic did not appear on the list of entrance requirements. Columbia College stated in its admission requirements that no one was to be admitted except those "expert in Arithmetic so far as the Rule of Reduction," and this was extended in 1786, to state that the applicant should understand the four first rules of Arithmetic, with the Rule of Three.[12]

Since mathematics was not classified among those subjects which contributed to the religious development of an individual, arithmetic was forced to make its début as a secondary school subject not in the grammar schools but in the early writing schools. In the eighteenth century we find a tendency toward more general recognition of the value of arithmetic as a school subject; this was due partially to the rise of commercial interests and also to the tendency of the people to be more hospitable toward a subject not supported by tradition.[13] In a biography of Caleb Bingham in Barnard's *Journal of Education* we find that "scholars of Latin schools were to attend writing schools two hours forenoon or afternoon." During this same time it was thought best that children "should begin to learn arithmetic at eleven years of age."

Textbooks were not generally used until the latter part of the eighteenth century. The Master dictated the problems with no special directions or explanations except for the rule to be used, and then it was the pupil's task to solve the problem. If anyone chanced to learn the fundamentals of arithmetic during this period, credit had to be given to the native ability of the learner and not to the skill of the instructor. Warren Burton said "that simple addition was easy 'but there was one thing that I did not understand—that carrying of tens. It was absolutely necessary, I perceived, in order to get the right answer, yet was a mystery which that arithmetical oracle, our schoolmaster, did not see fit

[12] E. C. Broome, *A Historical and Critical Discussion of College Admission Requirements*, The Macmillan Co., New York (1903), pp. 30–34.

[13] E. E. Brown, *op. cit.*, p. 134.

to explain.' "[14] Deacon Joseph Hawes, who went to Yarmouth just before the Revolution, gives the following picture of the methods used in the teaching of arithmetic in his school days:

> The most forward in arithmetic might do one or two sums within a day, if they could do them without the master's assistance; he gave me one sum in the single rule of three which I could not resolve for two or three days. After requesting him a number of times to inform me; he would reply that he had no time and that I must study for the answer.[15]

Under such a system of instruction the only mathematics studied was "ciphering" which consisted largely of a training in the manipulation of integers. In fact "a majority of the district-school pupils, especially the girls ciphered only through the four fundamental rules, with a short excursion into vulgar fractions. If they penetrated into the mysteries of the rule of three, they were accounted mathematicians; and if in later and degenerate days one ciphered through Old Pike (Pike's *New and Complete System of Arithmetic*) he was accounted a prodigy; and if he aspired to teach all the doors were open to him."[16]

The majority of arithmetics published during the colonial period were English editions. In the period from 1662 to 1776 there were only twenty-seven arithmetics published, and nine of these contained a great deal of material foreign to the subject. *Arithmetic: or the Ground of Artes*, published by Robert Recorde in first edition about 1540 to 1542, while not the first English text published, was one of the most influential and exerted a dominating influence over the teaching profession for at least a century afterwards.[17] This book consisted of three parts and was written in the form of a dialogue between "Master and Scholar." In Part One the author considered the four fundamental operations with whole numbers, reduction, progression, and proportion. In Part Two he discussed fellowship and further extended the treatment of the four fundamental operations with integers, he also included some work in bookkeeping and fractions. Part Three was composed of "Rules of

[14] F. Cajori, *The Teaching and History of Mathematics in the United States*, Government Printing Office, Washington, D. C. (1890), p. 53.

[15] W. H. Small, *Early New England Schools*, Ginn and Co., Boston (1914), p. 363.

[16] G. H. Martin, *Evolution of the Massachusetts Public School System*, D. Appleton and Co., New York (1894) p. 105.

[17] D. E. Smith, *Source Book in Mathematics*, McGraw-Hill Co., New York (1927), p. 13.

Practice, Questions of Factorage, Questions of Interests, The Golden Rule of Three, Loss and Gain, Rules of Payments, Questions of Buying, Tares and Allowances, Loan and Interest, Factorship, Bartering, Exchange, Monies, Weights and Measures, Sports and Pastimes done by Numbers, Plane Figurate Numbers, The Extraction of Square Root, The Extraction of Cubick Root, Tables of Board and Timber Measure, Interest, Annuitie Respited, in Present and Reversion."[18] These early arithmetics were written primarily from the utilitarian point of view. Wingate, the author of *The Arithmetic, Containing a Plain and Familiar Method for Attaining the Knowledge and Common Practices of Common Arithmetic* which was published in 1689, stated that the purpose of his arithmetic was "for the Ease and Benefit of such Learners, who desire only so much Skill in Arithmetick, as is useful in Accompts, Trades, and such like ordinary Employments."[19] This book differed very little from that of Recorde's and these differences were primarily in terminology and the fact that Wingate wrote his book in the third person.[20]

Probably the best of these early treatises on arithmetic, and certainly one of the most popular texts of the time, was *The Secretary's Guide* or *The Young Man's Companion* edited by William Bradford in 1705. This book, which went through three editions by 1719, was organized into four parts as follows: (I) Spelling, reading, and "writing true English"; (II) "Arithmetic made easie, the Rules thereof explained and made familiar to the capacity of those that desire to learn"; (III) forms of letters; (IV) "Bills, Bonds, Letters of Attorney, . . . Bills of Exchange, and other legal questions."[21] Another arithmetic written by an American author and edited in this country during this early period was that of Professor Isaac Greenwood of Howard College. This book, *An Arithmetick, Vulgar and Decimal: with the Applications therof to a Variety of Cases in Trade and Commerce,* was published in 1729, but found little use or circulation except in the author's own college classes.[22] Two

[18] H. Kimmel, "Status of Mathematics and Mathematical Instruction During the Colonial Period," *School and Society*, Vol. 9, p. 200.

[19] D. E. Smith, "The Development of the American Arithmetic," *Educational Review*, Vol. 52, p. 111.

[20] H. Kimmel, *loc. cit.*

[21] L. C. Karpinski, "The First Arithmetic in the United States" *School and Society*, Vol. 19, p. 349.

[22] F. Cajori, *op. cit.*, p. 14.

texts of foreign authorship that become so popular that they were later reprinted in this country were "Hodder's Arithmetick," the American edition from the twenty-fifth English edition appeared in Boston in 1719, and the English translation of Pieter Venema's "Coffer Konst," published in New York in 1730.[23]

Other texts published by American authors were: (1) *The American Tutor's Assistant* by Zachariah Jess (1788); (2) *A New and Complete System of Arithmetic* by Nicolas Pike (1788); (3) *The Schoolmaster's Assistant* by Nathan Daboll (1799); (4) *A New System of Mercantile Arithmetic* by Michael Walsh (1800); (5) *Scholar's Arithmetic* by Daniel Adams (1801); (6) *Scholar's Arithmetic* by Jacob Willett (1817). Adam's book went through nine editions and 40,000 copies were sold of the 1815 revision. Willett's book went through fifty editions within a short time and there was one revision as late as 1849. The two most popular of these books were those of Nicolas Pike and Nathan Daboll. The latter went through a revision as late as 1839. The second edition (1797) of Pike's *New and Complete System of Arithmetic*, which was used in colleges as well as the secondary schools, contained 516 pages divided as follows: 16 pages devoted to Recommendations, Prefaces, Table of Contents, and Explanation of Characters; 396 to arithmetic; 4 to plane geometry; 11 to plane trigonometry; 46 to mensuration of surfaces and solids; 33 to "an introduction to algebra designed for the use of Academies"; and 10 pages to an introduction to conic sections.

Probably the most popular text of this early period was *The Schoolmaster's Assistant* by Thomas Dilworth, first published in England in 1743. An American reprint of this book appeared in Philadelphia in 1769 and it later enjoyed at least six American revisions.[24] In the treatment of the subject matter no attempt was made to present any formal demonstration, and all rules and definitions were given in the form of questions and answers. As was characteristic of all texts of the colonial period, this book placed great stress upon the memorizing of rules. Cajori in commenting upon the book says "the whole book is nothing but a Pandora's box of disconnected rules. It appeals to the memory exclusively and completely ignores the existence of reasoning powers in the mind of

[23] *Ibid.*, p. 13.
[24] *Ibid.*, p. 14.

the learner."[25] By the methods of colonial times all problems were solved as being one of a certain type or as belonging to a particular case. Dilworth gave the process of multiplication in five cases and long division in three cases, and the reduction of fractions was given in twelve cases. Few pupils were able to solve a problem unless they knew under which case it came. As an illustration of the results of this method of teaching Cajori quotes from an article, "Early School Days," by Barnabas C. Hobbs. In remarking on a new system of teacher's examinations in Indiana, Hobbs says:

> I shall not forget my first experience with the new system. The only question asked me at my first examination was "what is the product of 25 cents by 25 cents?" . . . We were not as exact then as people are now. We had only Pike's Arithmetic which gave the sums and the rules. These were considered enough at that day. How could I tell the product of 25 cents by 25 cents when such a problem could not be found in the book? The examiner thought it was six and one-fourth cents, but was not sure. I thought just as he did but this looked too small for both of us. We discussed the merits for an hour or so, when he decided that he was sure that I was qualified to teach school and a first class certificate was given me.[26]

This system and these methods of attack were applied to problems on a variety of subjects. In notation and numeration, Dilworth gave rules for numbers up to nine digits. Pike extended these rules to any number of digits, giving a specific example for a number of forty-two digits. While Dilworth treated of the fundamental operations in a number of cases, several other texts presented this material in a much more simplified form. In common, or vulgar, fractions Dilworth presented the different operations as special cases, but did not solve an example or illustrate a rule as did Cocker and Hodder. Under the discussion of decimal fractions he included, in addition to the fundamental processes, the rule of three, interest, discount, equation of payments, and a number of other applications of percentage. The subject of denominate numbers was considered quite important, and Dilworth gave tables for English money; Troy, avoirdupois, and apothecaries' weight; time and motion. Percentage included problems of loss and gain, discount and interest, while Pike added commission, brokerage, partial payments, buying and selling stock, and policies of insurance. In addition Pike and Dilworth gave an elementary treatment of progressions and permutations, they also discussed

[25] *Ibid.*, p. 16.
[26] F. Cajori, *loc. cit.*

such topics as square root, cube root, barter, fellowship, alligation, practice, and position. Pike devoted thirty pages to a group of miscellaneous questions and problems, each problem, or group of problems, being preceded by a rule for the solution. A typical rule is the one for finding two numbers, having given the sum of the numbers and the sum of their squares:

> From the square of their sum take the sum of their squares: Then from the sum of their squares take this remainder, and the square root of the difference will be the difference of the numbers. To half their sum add half their difference, and the sum will be the greater. From half their sum take half their difference, and the remainder will be the less.[27]

Although textbooks came into rather general use during the latter part of the eighteenth and the early nineteenth century, the cipher book found a place in the system of instruction up until about the middle of the nineteenth century. Breslich describes the cipher books "containing the work of two pupils who studied arithmetic in Illinois in the years from 1804 to 1808." The topical outline as found by Breslich was as follows:

> (1) Interest, simple and compound; (2) rebate or discount, commission; (3) partnership; (4) fellowship and compound fellowship; (5) promissory notes, brokerage; (6) barter; (7) tare, trett, and cloff (allowance made to buyer for weight of box, bag, or container); (8) loss and gain; (9) insurance; (10) exchange; (11) annuities; (12) equation of payments; (13) rules for the solution of areas of triangles, squares, rectangles, and circles; (14) square root and cube root; (15) volumes of prisms, pyramids, cones, cylinders, spheres, and frustums of pyramids and cones; (16) compound multiplication; (17) measures and weights; (18) reductions; (19) arithmetical and geometrical progression; and, (20) rule of three.[28]

Although the range of subject matter prompts the conclusion that these cipher books were made by students pursuing a college course, Breslich observes that "the subject taught was but little more than a mechanical manipulation of figures and a study of rules dogmatically applied."

A point of interest, though of no particular pedagogical significance, that was generally characteristic of the cipher books, and also some texts, was the poetic form of stating rules and problems. The following is a portion of a poem twenty lines long giving

[27] Nicolas Pike, *A New and Complete System of Arithmetic*, Isaiah Thomas, Worcester, Mass. (1797), p. 360.

[28] E. R. Breslich, "Arithmetic One Hundred Years Ago," *Elementary School Journal*, vol. 25, p. 672.

the rule for extracting cube root. It was labeled "A Rule to Get by Head":

> The cube root of your first period take
> And of its root a quotient make.
> Which root into a cube must go
> And from your period taken fro.
> To the remainder then you must
> Bring down another period just, . . .[29]

An interesting example of a problem stated in this poetic form is the following application of the Pythagorean theorem:

> A castle wall there was whose height was found
> To be 100 feet from top to ground.
> Against the wall a ladder stood upright,
> Of the same length as the castle was in height.
> A waggish youth did the ladder slide
> (The bottom of it) ten feet from the side
> Now I would know how far the top did fall
> By pulling the ladder from the wall?[30]

All of this served to enliven and add some zest to what must have been a rather uninviting study, a study of rules dogmatically applied.

Arithmetic, the only mathematical study of importance in the secondary schools of the United States during the colonial period, had to gradually work its way into the curriculum of the Latin grammar school, and this was accomplished largely through popular demand and not through legislation. The subject matter was logical in arrangement and consisted primarily of a series of rules to be memorized and dogmatically applied according to rather definite classifications. No attempt was made to adapt the instruction to the pupil and very little change was made in the nature or content of mathematics in the secondary school during this early period. This whole system of instruction is rather vividly pictured in the following quotation from the preface to the *American Accomptant*:

> If he (the Pupil) is taught any of the rules to memory, he learns them like a parrot, without any knowledge of their reason or application. After this manner he gropes along from rule to rule, until he ends his blind career with the rule of three; and in the end, the only truest account he can give of the whole is that he has been

[29] *Ibid.*, p. 668.
[30] *Ibid.*, p. 669.

over it. But he has completed his school education and is well qualified to teach a school himself the next winter after.

This passive condition existed until after the introduction of the Pestalozzian movement into America, the spirit of which was expressed by Warren Colburn in his address before the American Institute of Instruction in 1830. This address marked the beginning of a new epoch in arithmetic and the teaching of arithmetic the full significance of which can be realized only through a study of the teaching of mathematics during the period of the academy.

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Weaknesses of High School Students Who Enter College Mathematics and a Suggested Remedy*

By INA E. HOLROYD

Kansas State College, Manhattan, Kansas

THE SUBJECT assigned me was "Weaknesses of High-School Students Who Enter College Mathematics and a Suggested Remedy." I should like to change this to read "A Few of the Weaknesses . . . and Suggested Remedies." There are several remedies for the situation in which we find ourselves. One of the most important I shall venture to suggest before discussing the main topic.

There is probably no one thing that would raise the standards of the mathematical output of our high schools and return to the public so rich a dividend as higher standards for mathematics teachers. Probably teachers of other subjects would not agree with me, but I think mathematics teachers will, that no subject in the high school curriculum suffers more from bad teaching technique than the subject of mathematics; and no subject in the curriculum might be made of more lasting and practical benefit. Mathematics classes are parceled out to whoever may happen to have a vacant hour because "mathematics is a sort of necessary evil required by colleges for entrance, but which no one ever uses." How any educated person in this age of science can possibly overlook the rôle mathematics has played, and is playing in the progress of our civilization is beyond comprehension.

I believe, however, that to a very large extent improvement must come from within the ranks of our own profession. Mathematicians must interest themselves in the elementary field sufficiently to give of their time and talents for the general welfare of the secondary schools. For this we owe a debt of gratitude to such men as E. H. Moore, E. R. Hedrick, the late J. W. Young, and a few others; but we need *many* more such men.

Our country as a whole is far behind many of the foreign coun-

* Read before the Mathematics Round Table, Hutchinson Section, Kansas State Teachers Association, Nov. 4, 1932.

tries in teacher requirements. Among the nations of the world, Sweden, Denmark, France, and Germany certainly maintain the highest standards in the selection of mathematics teachers for secondary schools. No small part of this is the remarkable training which these teachers get from the master minds and master teachers in the schools themselves. It is probably no exaggeration to state that Sweden leads the world in the extent of scientific preparation required for her teachers in the secondary schools.

Sweden's secondary school period is nine years, based on three years primary, closing with an examination which is the sole means of entrance to the universities. The courses for secondary-school teachers, which are very elaborate, are divided into two four-year periods, with an examination at the end of each period. Candidates for positions in the first five years of the secondary schools are required to pass the first of these examinations. Those seeking positions in the last four years must pass the second, and in addition must defend a thesis for the degree of doctor. In each case the candidate has to spend a probationary year in connection with one of the five special schools which organize excellent professional training. Moreover, before permanent appointment in a secondary school, the candidate must have spent at least two years in successfully performing the duties attached to the position.

Mathematical requirements in Denmark and The Netherlands are equally severe. Denmark requires two years as a probationary period before examination as to eligibility to teach; but The Netherlands does not require professional training. Professional training in France is meager, but mathematical requirements are unparalleled in any other country.

In all four of these countries, it is from among those who have passed the examination for the degree of doctor, that teachers for the secondary schools are almost wholly recruited.

Germany (primary education three years and secondary, nine years) insists much less than France on brilliancy and breadth of mathematical attainment, but stresses both theoretical and practical professional training. The candidate for a position in secondary schools must pass an examination the requirements for which are about our equivalent for the master's degree. In addition he must submit a thesis. Having passed, he must still undergo two years of severe professional training. Throughout the country the

attractions of the teaching profession are sufficiently strong to draw some of the best mathematical talent of the country.

To get a comparison: Since the future mathematics teacher *entering* a German university is about on a scholastic par with a student who has finished the *junior* year at a college in the United States, we may say, roughly, that a German teacher has generally had at least three years more scientific training than the prospective teacher going out as a college graduate in the United States. Moreover, in contrast, his training has as a rule been based upon a secondary education relentlessly thorough, and conducted by those who are masters of their profession. Such specialization is characteristic in most of the European countries in connection with the preparation of their mathematics teachers for secondary schools.

Now I do not wish to be understood as advocating that we superimpose the requirements of these countries upon our own system. They are probably not entirely adapted to our ideals as a democratic form of government; but I do insist that for our primary teachers, i.e., the first six-year period, the bachelor's degree from a standard college or university, including as a minimum mathematics through the analytics, should be required; and for the secondary-school teacher, i.e., the last six-year period, the master's degree in the subject to be taught, together with at least a year's professional training in that subject.

The first requirement of a good mathematics teacher is that he should know his subject matter, and enough of it to give ample perspective to the immediate courses which he is teaching. Next to this is thorough training in *how* pupils learn, and methods of presentation of certain topics so that the learning will be effective, i.e., will get well over the threshold of understanding. I care not how much a teacher may know about "γ functions" if he cannot impart his knowledge to the pupil, he is a failure as a teacher. Add to these requirements a sincere interest in the subject, his profession, and his pupils' welfare, and we may trust the results.

Let us now turn to a few of the most outstanding pupil weaknesses, and suggest perhaps some remedies.

Careful observation and analysis will show that the greatest single source of difficulty in mathematics is the lack of good study habits; and not, as is generally assumed, the subject matter or the inability of the pupil to understand it. The remedy is evident—

teachers *must* learn to give instruction in ways of acquiring effective study habits. This is quite as necessary as acquiring subject-matter—in a sense, more so. I refer you to the chapter on "Learning and Its Supervision," in *Psychology of Education*, by Judd.

In addition to habits of study which are essential to success in every subject, others have to be established which are especially valuable in the study of mathematics. The following will suffice as an illustration.

(1) The habit of careful reading—weighing meanings, scrutinizing relationships, comprehending in terms of one's *own* words.

(2) The habit of translation from verbal statements into symbols and, quite as important, from symbols into verbal statements. "Words are tools of abstraction."

(3) The habit of connecting words and symbols with meanings. Without this habit, pupils become manipulators, employing certain words and principles, not even attempting to comprehend what is meant; so much so as to deceive the very elect (even themselves) as to how much they really understand.

(4) The habit of making mental summary of what is known or given in a discussion and of what is to be proved or found.

(5) The habit of making drawings representing a situation, i.e., to make an abstraction concrete.

(6) The habit of estimating the approximate answer and of at least deciding whether the answer found is reasonable.

(7) The habit of checking results.

Fundamental principles should be thoroughly taught by means of *simple problems;* and trust to the pupils' general mental maturity and subsequent teachers to make applications in more complicated situations in later courses. Something is radically wrong when a college algebra class must spend half of the semester repeating the algebra of the high school. When pupils do not know the most fundamental principles, it is difficult indeed in an advanced course to fix up deficiencies, because the pupil has passed beyond these experiences and becomes impatient and bored when held to them—he is not psychologically in a receptive frame of mind. Many, I might almost say most, of the students I get in college algebra do not know when a fraction has been multiplied. They do not know that the fundamental principle of all fraction work is the fact that both numerator and denominator of a fraction may be multiplied or divided by the same number, not zero, without changing the

value of the fraction, and moreover that these are the *only* operations which do not change its value. They do not know that to divide the denominator of a fraction multiplies it as well as to multiply the numerator, nor do they know that to multiply a denominator will divide the fraction as well as to divide the numerator—in fact they do not even divide the numerator. The sum total of their knowledge here is bound up in the jingle—"Invert the divisor and cancel." They tell me that cancel means "to strike out—" they do not know the principle which lies behind this magic. I have long since utterly banished the word "cancel" from my classroom. It is one of the most pernicious words in the vocabulary of elementary mathematics. If it means to divide, then say "divide": if it means to destroy as $+a$ and $-a$, then by all means let us say "destroy" or "annul."

Chiefly as a result of all this, they tell me that the terms of an equation may be multiplied by the same number without changing its value. Where "its" is a most indefinite pronoun. Hence the magic of "clearing the equation of fractions." To them it is to "divide each denominator into the L.C.D. and multiply the numerator by the result." That the real performance is that every term of the equation has been actually *multiplied* by the L.C.D., but that the ratio of the coefficients has not been destroyed and consequently the value of the unknown unaltered has never passed over the threshold of understanding. Indeed they do not know what is meant by an unknown.

Or again, they may reduce the equation to the L.C.D. and either leave it under the equation or "drop" it. They do not know what has been done, but it works. As a result they will, when given fractions to add, especially if they contain an "x," reduce to the L.C.D., "drop" the denominator, equate to zero, and behold an equation!

Part of this confusion is a "hang over" from the methods of the average grade teacher of arithmetic employed in the operations especially of multiplication and of addition of fractions. Multiplication of fractions, say $5/6\times6$, or 12, or any other multiple of the denominator should not be taught $5/6\times12=10$; but $5/6\times12=5\times12/6$, then—"Dividing both numerator and denominator of the fraction by 6 and multiplying the factors gives 10." Adding $2/3+3/4$ should never be taught as follows: "The L.C.D. is 12, divide the L.C.D. (12) by the denominator of the fraction (3)

and multiply the numerator (2) by the result (4) thus giving 8/12, because it obscures the most fundamental principle of all fraction work. Children should be taught to say and to *think*—"I must multiply both numerator and denominator of the 2/3 by 4, which does not alter the value of the fraction, but reduces it to the denomination 12th." I.e., $2/3 \times 4/4 = 8/12$.

An accumulation of such details (not corrected by the high-school teacher) is a most fruitful cause of a large number of the difficulties which students have in college mathematics. As one instructor said, "I may use one sentence to summarize the entire field of difficulty which students have with algebra; namely, the average student fails to realize that algebra is generalized arithmetic." It deals with the properties of numbers, and not knowing the principles of arithmetic, how can the pupil generalize? "Any student will admit the truth of the following arithmetic:

$$10 - 7 = 3; \qquad 10 - (5 + 2) = 3; \qquad 10 - 5 - 2 = 3;$$

but is not able to see the analogy between the arithmetic and the algebra example $x-(y+z)=x-y-z$."

This emerges as another of the so-called "Recurring Errors—" namely, difficulty with minus sign before the fraction bar, both in subtraction of fractions and in clearing the equation of fractions. The pupil does not think of the fraction bar as a sign of aggregation, and the sign in front as a subtraction sign.

Again if 17 is divided by 5, the quotient is 3 and the remainder is 2. He will agree that $17/5=3+2/5$. But if $x-8$ be divided by $x-4$ to give a quotient of 3 and a remainder of 2, he writes $x-8/x-4=3+2$.

When it comes to dealing with a formula, the pupil does not realize that a formula is the answer (in a generalized form) to some certain type of problem. Pupils should be trained to make formulas for themselves in order to remedy this difficulty. "What is the cost of any number of articles at so much per article?" Such problems should occupy their attention for a considerable period of time in the early part of their algebra work, and the literal forms evaluated with the signless numbers of arithmetic.

The remedy for nearly all of these difficulties lies in taking the utmost care to clear every concept carefully as we come to it; to make haste very slowly in all beginning work; and to go over these again and again in new situations. But let me repeat— teach

principles by means of simple problems, and with all your teaching, teach principles. Do not be deluded by the fact that pupils get a correct answer—they are expert jugglers and may not have the slightest idea of what they are actually doing—question them in personal conference—let them tell you in words what they are doing and why. You will be quite surprised that they do not know as much as they appear to know.

Teach your equations as a thought process (not by axiom in the beginning work), employing simple examples with the signless numbers of arithmetic: $x+3=5$, $2x-7=3$, etc. Transposition is too good a tool to discard, but a dangerous one to use in early work.

Speaking of transposition, the following is a very common error with students: $5x=4$, $x=-1$ or possibly $x=-4/5$ if not $-5/4$. Any quantity which seems to change sides in an equation must have its sign changed, so why not! This is the result of failure to have it impressed upon the pupil that all conditional equations finally come to the form $ax=b$, that ax means $a\times x$ and that b represents the product. Knowing one factor of this product we find the other by dividing it by the known factor. They may not know the difference between factor and term either, but that is another story.

Keep the work simple, and do not be deluded into thinking that every problem the textbook offers must be solved—probably the pupil will never see anything like some of these problems again in in the whole course of his mathematics, and possibly it is to be hoped he never will.

I must say a few words about the teaching of positive and negative numbers. Most books and many teachers introduce directed numbers too early in the elementary course. The pupil has quite enough to do to become familiar with the symbols and their use in his early algebra work. Show him how to simplify arithmetical computation by means of some of the algebraic forms: $a^2-b^2=(a+b)(a-b)$; $ab+ac=a(b+c)$ etc. using the *signless* numbers of arithmetic. Also employ his time with such forms of the equation as he can do without *signed* numbers, until he comprehends his symbolism. It is then quite time for directed numbers. As Lincoln said to Seward when the latter was about to provoke war with England: "One war at a time, Seward, one war at a time."

I have practically never found a student in my algebra classes who understood the significance of positive and negative as used

in mathematics. Almost without exception they think the positive numbers are the numbers of arithmetic and that the negatives are the other end of the scale added backwards, when they began the study of algebra.

Now as a matter of fact, mathematicians until the beginning of the 17th century dealt exclusively with signless (or scalar) numbers The arithmetic scale is a signless scale beginning with zero and reading from left to right in an endless succession of numbers where each number on the right is one greater than the one preceding.

The directed (or vector) number is itself a product of the development of algebra. The algebra scale is a scale without beginning and without end, reading continuously from left to right, with each number on the right one greater than the preceding. It does not begin with zero and run in opposite directions. In point of significance of value, the minus symbols at the left of zero take whatever significance they have from their position relative to all the other numerals on the scale; not, as commonly supposed, because of their relation to zero. Positive numbers do not exist apart from negative numbers. This is no mere academic distinction. It goes to the root of the whole matter. It makes the operations of addition and subtraction of signed numbers rational, and instead of following a rule, the rule becomes the inevitable and necessary consequence of the meaning of positive and negative as "oppositeness of direction" from any arbitrarily chosen starting point called the "zero point." Thus writing a portion of the scale say from -9 to $+9$,

$$\cdots -9, -8, -7, -6, -5, -4, -3, -2, -1,$$

$$0, +1, +2, +3, +4, +5, +6, +7, +8, +9 \cdots.$$

To add:

$+5$	-5	$+5$	-5
$+2$	-2	-2	$+2$
$+7$	-7	$+3$	-3

Beginning with either number, count as many places, and in the *direction* indicated by the other number and its sign.

To subtract: Every subtraction problem is in reality an addition in that it asks the question, "What number must I add to my subtrahend to make my minuend?" This forces us to count *from* the subtrahend *to* the minuend, and never in any other direction. (Rising and falling temperature is a good illustration):

$+5$	-5	$+5$	-5
$+2\uparrow$	$-2\uparrow$	$-2\uparrow$	$+2\uparrow$
$+3$	-3	$+7$	-7

Beginning with the subtrahend, count *to* the minuend, giving the sign of the direction. We may use many devices to illustrate, but after all no better device can be used for teaching addition and subtraction of signed numbers than the scale. Let pupils count, count, count, until they are ready to take the short cut and formulate their own so-called rules. This is the road the race traveled, and I think we might get many such hints for our teaching from the history of the development of our mathematics.

To summarize: The high-school teacher, both junior and senior should

(1) Give continual attention to establishing careful reading and study habits.

(2) Keep up an almost constant review of arithmetical principles, especially in fraction work where pupils are so weak; and seek to generalize those prinicples in the symbols of algebra.

(3) Be sure that the meanings of the symbolism of algebra are perfectly clear before starting work with directed numbers. With these keep the idea of "oppositeness of direction" uppermost in the pupil's mind.

(4) Remember that translation from words to symbols and vice versa can scarcely be overdone.

(5) Proceed very slowly in the elementary work of algebra. You will gain the time. Clear every concept as you come to it.

(6) Teach principles, and how to think mathematically.

(7) Always establish the "why" of every operation.

(8) Beware of all mechanistic devices. Teach the thought process instead.

Let me say in closing that there are a few very fine books on these subjects which every teacher should own. Among those likely to be most useful to you are the following:

1. *The Teaching of Mathematics in Secondary Schools* (*Technique*) and *Problems in Teaching Secondary School Mathematics* (2 vols., $5.00), by Breslich: University Press, University of Chicago.

2. *The Fundamental Skills of Algebra* ($1.75), by Everett: Bureau of Publications Teachers College: 525 West 120th Street, New York City.

3. *The Teaching of Elementary Algebra* ($1.25), by Durell: Open Court Publishing Co., Chicago.

4. *The Teaching of Junior High School Mathematics* ($2.00), by Smith and Reeve: Ginn & Co.

5. *The Teaching of Secondary Mathematics* ($2.50), by Hassler & Smith: Macmillan Co.

6. *The Teaching of Algebra (Including Trigonometry)* by T. Percy Nunn, Longmans Green & Co. This should be in the library of every mathematics department.

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How is it Possible to Make Elementary Algebra More Real to the Pupil?

By ALBERTA S. WANENMACHER
Hutchinson-Central High School, Buffalo, N. Y.

THOSE OF YOU who have read the article by Judd in the April 1932 number of *The Mathematics Teacher* will remember no doubt that he accuses us of not always being efficient in our work. To the question, why students fail in mathematics, he received vague and general answers from mathematics teachers, many of whom were inclined to place the blame for failure entirely upon the students, calling large numbers of them mentally inferior. Judd then says very plainly that "either 50 percent of the world is not competent to follow in our footsteps or we're not competent to invite them, and we ought to find out which is the true description of the situation." Since Judd is not only a learned psychologist, but is also a firm believer in mathematical training, and a friend whose criticisms are constructive and aimed at us with the hope of awakening in us the urge to improve our teaching, it is important for us to ponder his words. His statistics in the same article show the continuous and rapid decrease in the number of pupils studying mathematics. Most of us are aware of this falling off in our own schools at the same time that school registration is increasing rapidly, and classes in some other subjects are growing by leaps and bounds. What are the reasons for this? Are teachers of other subjects gaining ground by being more appreciative of the pupils' efforts, more sympathetic with their problems than we, or are they better salesmen of their subjects? In the past teachers of commercial subjects have been able to advertise an attractive goal, namely a wage-earning job tied up with the high school diploma. Whether present economic conditions which have practically eliminated that goal will likewise cause a sudden decrease in commercial classes remains to be seen.

It is my desire to make a plea for good salesmanship, and to attempt, by showing how I have put into practice some of the ideas suggested by experienced educational leaders, to answer, at least partially the question, how is it possible to make elementary al-

gebra more real to the pupil. I feel that these two factors, good salesmanship and the making of elementary algebra more real to the pupil will, if carefully studied by every teacher of elementary mathematics, not only lessen by a marked degree the apathy, fears, and even failures in the required courses, but will also fill the vacant seats in the elective ones, for it is in elementary algebra usually where either a taste or distaste for mathematics is formed. I am not pleading for simplified algebra but for improved teaching which will really educate the student of mathematics.

Our task of salesmanship ought not to be a difficult one. We have a subject which is worth while (Judd says if it were taken from the curriculum "we would withdraw from our civilization one of the most important forces for mental development and training"), a subject which has stood the test of time, which the world needs, and in spite of all adverse criticisms let us believe with Reeve that "mathematics at its best needs no defense." We can go far in improving the teaching, in redirecting the emphases, in establishing ideals of what we want our courses to be, and in making a start toward those goals whether we can ever fully attain them or not. To make our salesmanship effective we have need of vision, a clearly defined goal toward which we are striving. We must believe in the future of mathematics, in the need of this training for our high school boys and girls that they may be better equipped to work out by mathematical methods the control of situations which will keep the world in a healthy, prosperous state. "Business as well as science needs men of brains who have a working knowledge of mathematics" says Edward Speare in "Mathematics in Pharmacy and in Allied Professions" in *The Sixth Yearbook of the National Council of Teachers of Mathematics*. We must prepare ourselves to answer understandingly the "whys," and "what it is all about," and reach the point where we are no longer afraid to encourage questioning. We must take time to find those pupils who should be encouraged to continue with advanced courses, explain to them the content of such courses, and guide them in making right choices. A year ago all but two of the girls in a third year mathematics class had been registered the previous year in the same sophomore studyroom. Nor were these girls by far all the girls in school eligible to such a class; in fact the studyroom from which they had come was the smallest of all the sophomore rooms. Why were these girls enrolled in the class and the others not? The

answer is simple, and it is this. The teacher in charge of that study-room is a mathematics teacher. She had watched the progress of the girls in their second year work, and when the time came to plan for the third year she selected these as being fitted for the course, explained it to them, and the advantages of the year course over the term of intermediate algebra. It was a simple case of salesmanship. Nor was the teacher wrong in her selections, for frequently, and without exception, the girls returned the next year to sing the praises of the class, and express their appreciation for having found it. Since the teachers who plan courses with the pupils cannot all be mathematics teachers, it should be the responsibility of each of us to take more time to explain future courses, to encourage those who are fitted for them and who will profit most to take them, and above all to make our teaching such that those who should go on are not hampered by an inadequate foundation or uninspired by dull teaching.

Making algebra real to the pupil is bound closely with understanding. A pupil solving countless problems by following meaningless rules, juggling empty symbols or copying model solutions is increasing understanding by not even the slightest degree, and can see no reality nor justification in the subject. If understanding is of first importance in making mathematics real to the pupil, what are essential factors in securing it? Breslich suggests three helpful facts to be kept constantly in mind by teachers in securing this understanding. The three are as follows: in the first place, "by teaching and reteaching, the meaning of the fundamental concepts of algebra employed in manipulative processes must be thoroughly known, Second, the pupil should constantly receive training in analyzing the relationships expressed by algebraic symbols, and third, pupils should be led to discover independently the laws of algebra."

The symbolism of algebra is meaningless unless concepts to which it refers are understood. This is possible only if they are based upon experiences. Bode in *Conflicting Psychologies of Learning* defines concepts as tools by which experienced situations are made over so as to take on new qualities. He says, "there is a sense in which we can say that the whole of education centers on the development of concepts." Judd tells us that pupils who are failing in algebra are not failing in all phases of the courses in mathematics, but they fail at certain points, and that if we can locate the points

at which they are failing we should have a key to a very large share of the difficulties. One of these strategic points seems to be the incomplete mastery of the mathematical concepts. Schorling in a study in 1925 listed sixty-three concepts for mastery in junior-high-school grades. Butler has selected thirteen of these as purely algebraic ones, and has listed under separate headings the concepts related to equality, measurement, graphs, and trigonometry. The list of the thirteen algebraic concepts as published in the March, 1932, MATHEMATICS TEACHER, is as follows: the root of an equation, the solution of an equation, coefficient, exponent, formula, positive and negative numbers, algebraic numbers, algebraic fractions, algebraic product, algebraic factor, a quadratic, a root of a number, and a power of a number. Butler says, "the mastery of concepts is indispensable to the development of the ability to do functional thinking which is one of the great aims of mathematical instruction." The pupils come to us from arithmetic where emphasis has been placed upon correct responses, procured to a large degree by habit training, where each particular question is concerned with a single result. They come with the capacity for learning to conceive abstractions, but not with the ability already formed. It is here that we must bridge the gap between concrete experiences and abstract thought, and in a much slower way, based first of all upon concrete experiences with which the pupils are familiar. Introduce abstract ideas, one at a time, and the meaning of algebra as a method of thinking, a subject which aims at recognizing relationships, and at making generalizations. Great care must be taken in the selection of these concrete experiences. for as Dewey says in *How We Think*, "if physical things used in teaching do not leave the mind illuminated with recognition of a meaning beyond themselves, the instruction that uses them is as abstract as that which doles out ready-made definitions and rules, for it distracts attention from ideas to mere physical excitations." "It is algebra which if rightly presented, increases the power of conception to an astounding degree, and liberates us from the concrete, visual, and tactual habits of thought that bind the naïve mind so closely to Mother Earth." This writer, a college professor of philosophy, says also, "the whole task of training the discursive reason, which lifts man above even the cleverest of other creatures, devolves upon the teacher of algebra." This last is probably true only when pupils do not receive training in geometry.

In developing the concept "formula," which is one of the earliest concepts to be introduced, let us aim consciously to avoid the manipulative side of formula work and stress "meaningful significance" which Everett designates "associative skills." Formulas should be summaries of pupils' experiences, results of investigations, and as far as possible formulated by the pupils themselves. When this is done the concept is based upon experience, it has meaning and no longer runs the risk of being translated into empty symbolism. The pupil has been active, and through this activity has gained insight. In which grade the first conscious attempt will be made to develop the concept "formula" will depend somewhat upon the school system. Whenever that may be, it is a new idea, which should arise naturally from a desire to shorten rules and processes, and be developed gradually by using various concrete devices. One method, among many, which I have found very simple and helpful is the use of a handful of toothpicks by each pupil. These he can lay on his desk to form equilateral polygons from which, after the concept perimeter has been reviewed, many perimeter formulas may be written. This follows the geometric approach for which Betz has listed at least ten advantages over any other. It fits also the statement which Betz says classroom experience has shown "that while the segment of a line as the simplest geometric figure should be made the principle tool for introducing the symbolism of algebra, only when segments are seen in their natural association with actual geometric figures do they really serve the desired purpose." This too conforms to the latest configurational psychology which recognizes wholes and analysis, instead of synthesis. We are told that we first see the composite figures, the squares, triangles, etc. and then analyze out the lines, instead of the reverse. Murphy of Columbia University in his article "Geometry of Mind," says, "experience is no mosaic of pieces, but in its original form a unit from which the process of living gradually splits off one part from another."

From these toothpick formulas the first real generalization in the formula $p=ns$ can be drawn. Practice in reading rules from these same formulas will give an understanding of numerical coefficient, another algebraic concept whose need arises early. For example, take the toothpick figure "hexagon" and move its sides so that they are placed in a straight line. Now the meaning of $6s$ as s added six times is made very clear. Instead of substituting nu-

merical values for s for the sole purpose of mere manipulation let them rather be used in studying the functional aspect from the formulas. Such questions as the following will do this. What happens to p as s increases from 2 to 3? from 3 to 4? etc. What happens to p if s is increased by 1? If s is doubled what happens to p? trebled? divided by 2? etc. The function concept is not an isolated topic but should permeate the whole of algebra. Breslich, says, "every unit should be planned to make definite contributions to the development of functional thinking." From all sides comes the plea to take more time for these worth-while things. Judd is concerned about it and says, "the teaching of mathematics moves too rapidly," and Breslich repeatedly stresses the mistake of not allowing enough time. "Maturation is one of the major conditions of learning" according to Wheeler in *The Science of Psychology*. Slow, gradual development of topics, punctuated with recess periods, during which certain differentiation processes take place, result in more complex responses upon the next presentation. The pupils trained to see in the formulas the existence of law, to recognize relationships, to think through to generalizations are the ones who become educated students of mathematics.

To my surprise when reading the article, "Developing a Concept of Proportion before Presenting the Formal Work," by T. L. Engel, in the March, 1932, number of *School Science and Mathematics*, I found the exact description of a method which we have used for a number of years. Engel says that since he has seen it used in the art galleries of Europe in copying pictures he has used it with success in his own classes. We have found that not only has it developed the concept "proportion" which is its primary purpose, but it has served admirably in furnishing interesting posters for display in the rooms and in display cases. Last February when we were celebrating the birthdays of Washington and Lincoln many pupils selected appropriate pictures and enlarged them by this scheme. In one room an ambitious pupil drew a large head of Washington on the graph board. The results which pupils of no special artistic ability are able to achieve by this method are surprising.

I have selected the teaching of simple linear equations to illustrate Breslich's second point in teaching for understanding, namely, "that the pupil should constantly receive training in analyzing the relationships expressed by algebraic symbols." Reeve points

out in *A Diagnostic Study of The Teaching Problems in High-School Mathematics*, that after thirteen lessons the percent of correct responses to $x+5=9$ in a test given to 1204 pupils was 97.5, while in the case of $1/3y+2=5$ the percent dropped to 64.3. While this study shows the use that can be made by the classroom teacher of similar tests for diagnostic purposes, and proposes many questions concerning the reasons for the results, it was not its purpose to reach definite decisions but rather to leave to future studies the opportunity to throw more light on some of the questions. One of these studies which does this, and which has influenced my teaching is *Fundamental Skills of Algebra* by John P. Everett. To each of you not already familiar with it, I recommend this study. Everett has an instructive article, "Algebra and Mental Perspective" in *The Seventh Yearbook of the National Council of Teachers of Mathematics*. In this article he says, "solving a problem by transposition or the use of axioms probably involves not even a vestige of appreciation of the reasoning which the equation is designed to further. There is no such thing as an algebraic equation in which the ideas of more and less are not included as well as the idea of equality. It is because of their failure to take more and less into account that solutions by transposition and axioms are inadequate." If the relationships expressed by the symbols are analyzed, and instead of interest centering entirely on the value of x, pupils are trained to see and read these relationships, the natural operation is associated with the result desired. Returning to the solution of the equation $x+5=9$, the first question about it which the teacher should ask is not what is the value of x, but rather what is the relationship expressed? The correct answer would be, 9 is 5 more than some number x. The next question would then be, 9 is 5 times more than what number? Of course the response to that would be that 9 is 5 more than 4, and therefore x equals 4. Since 4 increased by 5 equals 9 the result checks. Soon several pupils in the class are able to take the teacher's place in asking the questions, and the conversations about the equations can then be carried on by the pupils entirely until finally all pupils are able to ask and answer the questions for themselves. I have found it advisable to delay individual written work on this topic until the pupils have formed the habit of asking and answering such questions so that they become a natural part of every solution. This training in reading expressed relationships in simple equations is also an excellent introduction to verbal problems. It

isn't the solution of the equation which is the important thing, but the mental activity which is required in making comparisons. Everett says that when the pupil has seen these relationships the fundamental thinking of the problem has been done, and the solution is just an incident which follows naturally. From Betz we get the valuable suggestions of relating the equation to the formula and pictorial methods to aid in the understanding of these equations.

Breslich's third point which I quoted sometime ago, and which is a natural outgrowth of the second is "to lead the pupil to discover independently the laws of algebra." Here is where the teaching of algebra should move slowly, so that the pupil may have the time to reach for himself a definite goal. He must interpret many equations and return constantly to basic meanings before he will appreciate the fact that the indicated process should be undone by the use of the inverse process. Psychologists tell us that here lies an economy of learning, for the extra time spent in the slow development is later more than saved for the solution of equations is no longer "a bag of tricks," and pupils so trained are working on a very different plane from those who follow blindly a set of rules. Although the ideas of relationship are adequate for work with any linear equations, later axioms may be used to advantage and some of the earlier details discarded. This is safe only if frequent returns are made to these associative skills of relation. "Comprehension of the meaning of the axioms" says Everett, "is then made to grow out of, and be a part of the ability to generalize, which it is the peculiar function of algebra to develop."

Sometime very much later in the course when the problem of transforming formulas requires drill, I have used a competitive game which has proved of interest to all, and is especially helpful in keeping on the alert those pupils who are apt to be indifferent. The occasion of a period on a day when there is "something in the air" as for example a long or unusual assembly period, or before a vacation is a most opportune time to play the game. The class is divided into two groups of equal numbers. For convenience I shall call them here groups A and B. The first pupil of group A goes to the board. I dictate a formula and tell the letter for which it is to be solved, and the pupil writes the formula on the board. He then explains the relationships of the terms in the two members and what he is going to do to take the first step, and writes that step on the board. If his work is correct a second pupil from the

same group explains and writes the second step, and it is continued in this way by pupils of the same group as long as their work is correct or until the formula has been transformed. If an error is made the first pupil of group B repeats the step, the second group thereby getting the lead. The group given credit for the problem is the one which has the pupil taking correctly the last step. If group A was given the first step of formula one, then group B starts off number two, etc. In that way each group is given the same number of chances to start and also has the same number of opportunities to take the first step that being usually the most difficult. A chart is drawn on the board and a stroke made for each correctly completed solution. Usually one period is too short and the pupils (especially those of the group which is behind) beg to continue another day. In the meantime many of those who have made the errors come in during free periods or after school to practice, and it doesn't take very long before the game has served its purpose and is no longer any fun because all do well in transforming formulas.

I have spoken already of the need of complete mastery of algebraic concepts as one of the outstanding points in teaching for understanding and in making algebra more real to the pupil, of formulas for their early and continued use in developing functional thinking and their rôle in expressing the facts of the world about us in concise and precise language. Now comes to mind the familiar phrase from the *National Committee Report*, "the ability to analyze a problem, to formulate it mathematically, and to interpret the result should be dominant aims in the teaching of algebra." Since there is not time this afternoon to consider each of the central objectives common to all published lists in my attempt to answer the question how it is possible to make elementary algebra more real to the pupil, I shall conclude with a discussion of verbal problems. Their importance is well agreed upon for in addition to training in functional thinking they serve also to supply useful information from certain situations in applied problems, and in the solution of the equations formed as a source of drill in many of the algebraic processes.

Our first consideration is the attitude of the pupils toward problem solving. Right attitudes are all important, and the key to ease and understanding, and must be built up by well planned guidance. How are right attitudes toward problem solving to be acquired by

the pupils? I would give first place in importance to a gradual and thorough preparation for problem solving. This can be begun very early by translating simple verbal statements into symbols. Before the pupil can write even the simplest of formulas as $p = 3s$ referred to earlier from the toothpick figures, he has to have in mind the statement that the perimeter of an equilateral triangle is equal to the sum of the three sides. Here s is used to denote the number of units in the length of one side, and if in addition to its usage as an initial letter of the word side, the other and more important interpretation of general number is put upon it, the pupil is ready early to translate with meaning verbal statements into symbolic forms. All textbooks supply lists for such translations and many others may be made easily and quickly by the teacher.

Not only is the translation of verbal statements into symbols important training in preparation for solving problems, but the converse, that of translating symbolic statements into verbal ones in a variety of ways, is, I feel, equally important. Earlier I have referred to this translation as a means of understanding in solving equations. The learning of the many meanings which each operating symbol as $+$, $-$, $=$, etc. can have is excellent preparation for problems in that it trains the pupil to recognize mathematical words. As simple an equation as $3n+7=22$ has many translations, for example, 22 equals 7 more than 3 times a certain number n; 3 times a certain number n increased by 7 equals 22; the sum of 3 times a certain number n and 7 equals 22; 7 added to 3 times a certain number n equals 22; etc. As many variations for the symbol of equality can likewise be given. The pupils like this, it becomes a game, and they challenge each other to write the longest lists of different statements. Unless I am very much mistaken in both my own observations and in my reading of what others think about this, it all serves a very real need in training, in problem solving.

Learning the simple and much used formulas, and recognizing them and knowing them in their various forms, i.e, $d = rt; t = d/r = d/t$ is another important step in the gradual and thorough preparation which I feel is necessary in acquiring right attitudes toward problem solving

As a second factor in building up right attitudes I would emphasize the need of making careful selection of the problem material itself. In this connection I would call your attention to a study based upon statistical analysis made by Powell called *A Study of*

Problem Material in High School Algebra. In this study he gives five hundred problems classified as to genuineness, importance, and interest. "The chief aim of the study being," he says, "to compile a list of five hundred of the best problems in elementary algebra that are available at the present time" (1929). I will take time to quote a few of the conclusions of relative pupil importance which Powell has drawn from the analysis without making any comments on them.

1. Problems that deal with any kind of material advancement are given first choice as to importance.

2. The pupil is most interested in problems that express activity; that describe situations in which he may participate. A problem stated in terms of child activity receives higher evaluation by the pupil than the same problem expressed in terms of adult activity.

3. The pupil likes a problem better when it is stated so as to give a definite challenge. It makes little difference whether this challenge involves a puzzle or a genuine problem.

4. The pupil is not necessarily interested in problems because they are important. The problem of the teacher in this connection is one of determining some means of presenting this valuable material so that it will appeal to the pupil.

5. Too many difficult problems given at one time tend to lessen the interest of the pupil. The pupils however express a liking for difficult problems of certain types, if only a few are introduced at a time and if these are well distributed among the ones of less difficulty.

After a conscious attempt has been made to have pupils acquire right attitudes toward problem solving by a well-prepared approach and care exercised in the selection of interesting and worth while material, the task of the teacher is by no means completed. One teacher of my acquaintance, in the days of the old model-T Ford, used to say that her pupils needed self-starters. Being once started their major difficulties were over. Just what kind of self-starters are right and adequate has been a problem which I have been studying for a long time. A set of hard and fast rules that will work might be a boon to the weak student, or an elaborate technique a cynosure to the mentally lazy, but if verbal problems are to serve their primary purpose of teaching pupils to think through new relations, we must take care that our guidance right here is intelligent. Neither go too far nor on the other hand leave the pupil without some guide that will give him the ability to ask himself the right questions and confidence in his own powers.

First of all I tell my pupils that they need to read their problems as they read when first learning, emphasizing every word, and not

as they now read an interesting story. That each word and phrase must be clearly understood, and if not must be looked up for the meaning. They must put themselves into the situation or better reduce the same situation into terms of their own activity. We have class discussions on rewording problems, and reducing them to bare essentials, and individual assignments until the pupils feel this is a part of the solution of every problem. This later proves a case of "making haste slowly."

Why does a pupil succeed with guidance and flounder helplessly when left alone? Is it not that in the former situation he is led by stimulating questions? But of what avail unless he learns to ask himself these same questions. Should not then our attention at this stage of problem solving center upon training the pupils to ask themselves key questions? What questions? Probably the first ones should be, how many, and what are the situations in the problem? If problem solving in mathematics classes it so serve the purpose of preparing the pupils to solve better the problems arising in every day activities, groups of unclassified sets will provide the best training. A classification on the basis of mathematical relationships is provided by Breslich. This would group together problems of the form $x = ay$, which would include those whose relationships are expressed by some of the following formulas; $i = prt$; $A = bh$; $d = rt$; $c = pn$; etc. Although the pupil then knows that the relationship is to be expressed as the product of two quantities, he has more real thinking to do in selecting the quantities and their right relationships than he has when it is already classified under a certain type. However, Breslich claims, "the best practice for the teacher to follow is to group problems as frequently in mixed sets as by types. This retains the advantages of the case method and eliminates the disadvantages that may arise from it." The pupil then needs to ask himself the questions, what facts are known, and what are to be found? Clear and complete statements used in setting down these facts are important, while their relations to each other, and to the problem as a whole must not be lost sight of for a moment. A tabular arrangement of such statements is not to be recommended at the beginning at least, since brevity is its sole recommendation, and when used it tends to make the pupil satisfied with incomplete and obscure statements, and is a sort of automatic arrangement which makes the process mechanical. Often pupils taught by this method from the beginning are able to fill the boxes in a table

correctly, and with a fair amount of speed, and yet are unable to pick out the relationships needed to write the equation, for their whole attention and interest has been centered upon filling in a certain number of spaces without any thought about their relation to the whole. What relationships are expressed, and also what other relationships are implied are questions to be kept constantly at the focus of attention. Underlining phrases in the problem which state these relationships, or rewriting such phrases, later to be translated into symbols are good habits which require special training for their development.

"There is no royal road to learning," nor is there any one method of making algebra real to the pupils. What I have tried to give you this afternoon are a few of the points which I have found useful, and many quotations, which fit these points, from the writings of experienced and well known teachers.

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There is no imaginable mental felicity more serenly pure than suspended happy absorption in a mathematical problem.

—Christopher Morley in Plum Pudding.

Higher Standards for Teachers of Mathematics

By W. D. REEVE
Teachers College, Columbia University

THE MAJOR PART of the criticisms of mathematics are due to faulty and antiquated methods of teaching rather than to the inherent difficulty of the subject matter of mathematics itself. Mathematics, if properly taught, would be enjoyed by a large majority of the pupils who study it. Why else do we find so many adults who have gone through our secondary schools with a serious emotional complex against algebra? What can we expect when we reflect that in many localities the athletic coach with little training, if any, in mathematics is assigned to teach an algebra class on the theory that any body can teach algebra? Sometimes it even appears worse than that—someone is assigned to teach algebra because he happens to have a vacant period at the time the algebra class is scheduled.

There was a time when perhaps we should have been compelled to assign incompetent people to teach mathematics because we had nobody better available. Today such is not the case. We have an oversupply of teachers of mathematics, but we do not have an oversupply of good ones. Now is the time for The National Council of Teachers of Mathematics to use their influence to raise standards for teaching and see to it that at least new appointments to positions as teachers of mathematics be given to those who are best qualified to receive them. Moreover we should try to get administrators generally to see the importance of employing only the best candidates.

We shall not succeed in our attempt to raise standards generally if we are not willing to improve ourselves and if we do not set about to do it at once.

Improvement of teachers in service can be realized through teachers meetings, clubs, reading, summer school, and the like. If a teacher cannot attend school further, and does not belong to some live club, or cannot go to regular meetings of mathematics teachers his improvement is likely to depend upon his reading. The Mathematics Teacher and the National Council Yearbooks constitute the best single source of helpful material. Can we not make these sources better known to teachers of mathematics throughout the land by talking about them wherever mathematics teachers are gathered together?

Accuracy in Addition

By P. H. NYGAARD
North Central High School, Spokane, Washington

MUCH OF THE MATHEMATICS taught in the grades and in high school is being attacked on account of its presumed lack of value. Addition is one of the few parts of mathematics that is not being seriously criticized in this respect. Hence it is highly important that addition should be taught in such a way as not to jeopardize its usefulness.

The widespread use of calculating machines has made speed of no great importance in the teaching of addition. Speed is an item worth considering only when a person is to do a great deal of adding, but under such circumstances an adding machine would almost invariably be employed. When a person encounters an addition problem only once in a while, as is the case with most people, it makes little difference whether the total can be obtained in one minute or in two minutes, but it makes a world of difference whether the total obtained is right or not. For such incidental addition problems machines are not often available. Unless the person has ready at his command an ability to add such problems accurately, his arithmetic training is certainly defective.

Very few pupils who have finished their grade school work in arithmetic can be depended upon to add correctly a column of numbers having two or more digits each, especially if the column is quite long. The usual procedure taught in the grades is inadequate. It emphasizes speed at the sacrifice of accuracy. Many textbooks and many school systems train the pupils to use a method like this: Add the units' column down; write the last figure of the total in the units' place in the answer; do not write the number to be carried but remember what it is; add the tens' column down starting with the number carried; put the last figure of the total in the tens' place in the answer; carry mentally to the next column; add it down; and so on; then check after the whole problem is done by adding the columns up.

The writer recommends that pupils be taught the following method: Add the units' column down; write the last figure of the obtained total in the units' place in the answer; write the number

to be carried at the top of the tens' column, making it smaller and lighter than the other numbers; then add the units' column up to be sure that the total obtained was right; add down the tens' column starting with the number carried; write the last figure of that total in the tens' place in the answer; write the number to be carried lightly at the top of the hundreds' column; then add the tens' column up, ending with the number carried, to be sure the total was right; continue in this manner until all columns have been added. The process for the accompanying short column is as follows:

$$\begin{array}{r} {\scriptstyle 1\,2} \\ 928 \\ 737 \\ 854 \\ 418 \\ \hline 2937 \end{array}$$

Say 15, 19, 27; write 7 in the answer; write 2 above the tens' column; say 12, 19, 27 noting agreement; say 4, 7, 12, 13; write 3 in the answer; write 1 above the hundreds' column; say 6, 9, 11, 13 noting agreement; say 10, 17, 25, 29; write 29 in the answer; say 12, 19, 28, 29 noting agreement. If the total obtained by adding up does not agree with the other, then that column must be added down and up again until the totals agree before proceeding to the next column.

The two main points at issue between the methods are:

1. Should the numbers carried be written or simply kept in mind?

2. Should each column be checked before going to the next column or should the checking be done after all the columns have been added?

The writer has tried to find out why there is such a holy horror connected with writing the numbers that are to be carried. There seems to be a general feeling that it is not neat to write them. How the writing of a few small numbers can spoil the neatness of a paper is hard to see. Moreover most teachers place too much stress on neatness, just as if excellence in this respect could go a long way toward making up for wrong answers. Some teachers express the opinion that to write these numbers develops a bad or wasteful habit. If pupils do not have any worse habits than this when they leave school, the teachers can feel proud indeed. An argument is

also put forth by a few teachers that if these numbers are written at the top of a column they will be confused with the numbers that are a part of the original column. Such confusion can scarcely arise if the numbers are made, as suggested, smaller and fainter than the others.

The only reason that can be dug up in favor of checking after all the columns have been added is that the answer can be found more quickly this way. If there is time, the work can be checked; if there is not time, the answer can be left as a pretty good guess. Theoretically the checking is to be done afterwards, but in practice most teachers allow the checking to be entirely omitted. Speed tests rate a pupil according to the number of correct answers he can produce in a given time. In such tests a higher score can be obtained by not checking than by checking, because it takes a pupil as long to check a problem as it takes him to do a new problem. The use of such speed tests to rate pupil achievement and, indirectly, teaching efficiency makes teachers feel that they are justified in not demanding careful, accurate work. The viciousness of speed tests should be realized and their use discontinued.

The method here recommended of writing the numbers to be carried and checking each column before going to the next has many inherent advantages. Some of these will be enumerated.

1. An error made in adding the first column is given no chance to spoil the next column. Thus if a person gets 69 for the first column instead of 71 and does not check at once, he will carry a 6 instead of a 7 and get the second column wrong also.

2. If the adding is interrupted before it is finished either due to some external disturbance or due to lack of concentration on the part of the person doing the adding, only the column being done when the interruption occurred needs to be repeated. If the other method is used, it is necessary either to go back to the very beginning or to make a more or less uncertain guess as to what number was carried.

3. An error of 10 in the total of a column, which is an error quite frequently made, can be detected at once. With the other method its detection is very troublesome.

4. A problem involving many large numbers need not be done in one continuous mental effort. A rest can be taken after each column, if desired. To hesitate when the other method is used makes it likely that the number to be carried will be forgotten.

5. The checking becomes an integral part of the process, not an afterthought too often neglected.

6. A person using this method has a feeling of sureness, whereas the other method leaves him stumbling around in the dark.

Many schools and teachers are, no doubt, using the method whose advantages have been detailed above—or at any rate are using a method similar to it in all essential points. Those who are not would do well to consider carefully whether their particular method can be defended. Does it develop a habit of accuracy in that type of arithmetic work which the pupils will need in later life more than any other type?

Colin Maclaurin

Born at Kilmodan, Argyll, February, 1698
Died in York, June 14, 1746

COLIN MACLAURIN seems to have been a prodigy in mathematics. He entered the University of Glasgow at the age of eleven and took his masters degree at fifteen. At nineteen, he earned the post of professor of mathematics at Aberdeen by competitive examination. He spent several years in travel as tutor to an English family. In 1724 he won a prize offered by the French Academy of Sciences with an essay on the percussion of bodies. In 1740, he divided a similar prize with Euler and Daniel Bernoulli, his entry being an essay on tides. In 1725 he was made the deputy of the mathematics professor at Edinburgh and later became professor of mathematics there. The Edinburgh appointment was one in which Newton was greatly interested and Ball says that "Newton privately wrote offering to bear the cost so as to enable the university to secure the services of Maclaurin."*

Maclaurin took an active part against the Young Pretender in 1745 and when "Bonnie Prince Charlie" took Edinburgh, Maclaurin fled to York rather than to submit. His death was due to exposure in the fortifications about Edinburgh.

His *Geometrica Organica* (London, 1719 or 1720) dealt with conics, nodal cubics, quartics, and the general properties of curves.

* W. W. R. Ball, *A Short Account of the History of Mathematics*, 1915 ed., p. 384.

This was followed by *De Linearum Geometricarum Proprietatibus* (1720).

The publication of Berkeley's *Analyst* in 1734 stimulated Maclaurin to write a reply, but, feeling that Jurin was caring for the matter so far as its immediate exigencies demanded, he turned to the writing of a *Treatise of Fluxions* which appeared in 1742. In a life of Maclaurin published in one of his posthumous works, there appears this account:

> Mr. Maclaurin found it necessary to vindicate his favorite study, and repel an accusation in which he was most unjustly included. He began an answer to the bishop's book; but as he proceeded, so many discoveries, so many new theories and problems occurred to him, that, instead of a vindicatory pamphlet, his work came out a complete system of fluxions with their application to the most considerable problems in geometry and natural philosophy.*
>
> . . . Mr. Maclaurin found it necessary, in demonstrating the principles of fluxions, to reject altogether those exceptional terms (infinite and infinitesimal), and to suppose no other than finite determinable quantities such as Euclid treats of in his geometry."*

At least three-quarters of the work deals with the derivation of fluxions for various curves and with the topics of maxima and minima and curvature all treated by the classic geometric methods. In this treatise, Maclaurin showed that if a homogeneous liquid mass were to revolve uniformly about an axis under the influence of gravity, it must assume the form of an ellipsoid of revolution. It was this which Clairaut used a year later in his *Théorie de la figure de la Terre.* Maclaurin's work represented the high point of rigor in the exposition of the calculus in the eighteenth century, but it was not widely read as is indicated by the fact that the second edition did not appear until 1801.

Maclaurin's *Algebra*, published posthumously in 1748, was founded in part in Newton's *Arithmetica Universalis.* Mention was made in the January issue of this journal of the history of the theorem erroneously called "Maclaurin's Theorem."

Important as Maclaurin's work was in the history of the calculus his influence was unfortunate, for his emphasis on geometric methods caused British mathematicians in the next generations to avoid the methods of analysis and thus drop behind their contemporaries on the continent.

VERA SANFORD

* Florian Cajori, *A History of the Conceptions of Limits and Fluxions in Great Britain from Newton to Woodhouse.* Chicago, 1919, pp. 183–184.